The 15-Puzzle

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1 Introduction

This paper will consider a sliding square puzzle commonly known as the *15-Puzzle* in group theory. The puzzle is commonly, but incorrectly, attributed to Sam Loyd; An American chess player and puzzle author raised in New York City (O'Connor and Robertson,2003). The actual inventor, Noyes Chapman, applied for a patent in March 1880 in the midst of its greatest popularity (Slocum, 2023).

The 15 puzzle consists of 15 numbered squares all placed in a 4x4 box leaving one position empty. The un-shuffled initial position takes on the form:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

The challenge of the puzzle is to take some arbitrary board-state and return to the un-shuffled initial position. The challenge set by Sam Loyd, known as the *14-15 puzzle*, but will be known as *the Evil Puzzle* in this paper, comprised of simply swapping the initial positions of the 14 & 15 numbered squares. An Example of which is shown below:

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

Loyd offered $\$1,000$ to anyone who could solve it (O'Connor and Robertson, 2003). However, he conveniently did not mention that this particular board-state had been proven to be impossible in 1879 (Slocum, 2023).
The *15-Puzzle* can be algebraically captured by Group theory and will be explained throughout this paper in its full generality.

2 Definitions

Let $P = \{p_1, p_2, p_3, ..., p_{16}\}$ denote the set of all pieces in the puzzle, where p_i denotes the i'th numbered square. We let p_{16} denote the empty square. Therefore, the un-shuffled initial position would take the form:

p_1	p_2	p_3	p_4
p_5	p_6	p_7	p_8
p_9	p_{10}	p_{11}	p_{12}
p_{13}	p_{14}	p_{15}	p_{16}

Let $C = \{c_1, c_2, c_3, ..., c_{16}\}$ denote the set of all cells in the $4x4$ container in the following *"wiggly"* order:

c_4	c_3	c_2	c_1
c_5	c_6	c_7	c_8
c_{12}	c_{11}	c_{10}	c_9
c_{13}	c_{14}	c_{15}	c_{16}

Definition 2.1. A **snapshot** $s : P \to C$ is a bijection.

Definition 2.2. A **configuration** σ is a snapshot with $\sigma(p_{16}) = c_{16}$. So, in a configuration the blank is at the lower-right corner.

Suppose δ takes on the form:

p_2	p_1	p_3	p_4
p_5	p_6	p_7	p_{13}
p_{14}	p_{10}	p_{11}	p_{12}
p_8	p_9	p_{15}	p_{16}

Then, δ is a configuration since $\delta(p_{16}) = c_{16}$. This configuration is described by the permutation:

$$\delta = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ c_3 & c_4 & c_2 & c_1 & c_5 & c_6 & c_7 & c_{13} & c_{14} & c_{11} & c_{10} & c_9 & c_8 & c_{12} & c_{15} & c_{16} \end{pmatrix}$$

The un-shuffled initial position and the Evil puzzle are also configurations. Let α and β denote these configurations respectively. Then:

$$\alpha = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ c_4 & c_3 & c_2 & c_1 & c_5 & c_6 & c_7 & c_8 & c_{12} & c_{11} & c_{10} & c_9 & c_{13} & c_{14} & c_{15} & c_{16} \end{pmatrix}$$

$$\beta = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ c_4 & c_3 & c_2 & c_1 & c_5 & c_6 & c_7 & c_8 & c_{12} & c_{11} & c_{10} & c_9 & c_{13} & c_{15} & c_{14} & c_{16} \end{pmatrix}$$

An example of a snapshot that is not a configuration:
Suppose μ takes on the form:

p_2	p_1	p_3	p_4
p_5	p_6	p_7	p_{13}
p_{14}	p_{10}	p_{11}	p_{12}
p_8	p_9	p_{16}	p_{15}

Then clearly we do not have a configuration since $\mu(p_{16}) = c_{15}$.

Let S be the set of all snapshots.

Definition 2.3. A **basic move** m is a permutation in S_{16} such that $s_2 = m \circ s_1$ for some $s_1, s_2 \in S$ and m should be "legal" in the sense that going from s_1 to s_2 should be obtained by exchanging the blank space p_{16} with one of its neighbours by sliding the non-blank piece to the cell of the blank space (up,down,left,right whatever applies depending on the cell). A **move** M is a finite sequence of basic moves and so as a permutation it is given by the composition of the permutations defining the basic moves.

Let m_1, m_2, m_3, m_4 denote the basic moves $up, down, left, right$ respectively. The following example will clearly illustrate the four basic moves that will be used throughout the text.
Suppose we begin with the following setup s_1:

p_1	p_2	p_3	p_4
p_5	p_6	p_7	p_8
p_9	p_{10}	p_{11}	p_{12}
p_{13}	p_{14}	p_{16}	p_{15}

Then, suppose s_2 is of the form:

p_1	p_2	p_3	p_4
p_5	p_6	p_7	p_8
p_9	p_{11}	p_{14}	p_{12}
p_{13}	p_{10}	p_{16}	p_{15}

where $s_2 = M \circ s_1 = m_1 \circ m_3 \circ m_2 \circ m_4 \circ s_1$. Since all elements not including the subset $\{p_{10}, p_{11}, p_{14}, p_{16}\}$ were unaffected, it would be easier to look only at this portion of the total board and run through each basic move $m \in M$ that resulted in s_2.

$$\begin{array}{|c|c|}\hline p_{10} & p_{11} \\ \hline p_{14} & p_{16} \\ \hline \end{array} \overset{m_4}{\longrightarrow} \begin{array}{|c|c|}\hline p_{10} & p_{11} \\ \hline p_{16} & p_{14} \\ \hline \end{array} \overset{m_2}{\longrightarrow} \begin{array}{|c|c|}\hline p_{16} & p_{11} \\ \hline p_{10} & p_{14} \\ \hline \end{array} \overset{m_3}{\longrightarrow} \begin{array}{|c|c|}\hline p_{11} & p_{16} \\ \hline p_{10} & p_{14} \\ \hline \end{array} \overset{m_1}{\longrightarrow} \begin{array}{|c|c|}\hline p_{11} & p_{14} \\ \hline p_{10} & p_{16} \\ \hline \end{array}$$

Where $m_4 = (15, 14)$, $m_2 = (14, 11)$, $m_3 = (11, 10)$, $m_1 = (10, 15)$. Remember, the numbers in the permutations indicate the cells and not the numbers on the blocks themselves since $M : C \to C$.

Definition 2.4. Let s be a snapshot. Then we say $\sigma = \sigma_s$ is the **normalization** of s if σ can be obtained from s by a move moving the blank space, p_{16}, all the way to the c_{16} following the wiggly trajectory i.e. in each basic move in the composition of the move the index of the cell containing p_{16} increases by 1.

For example, consider the following snapshot s:

13	2	7	4
10	3	6	8
9		1	11
12	14	15	5

To go from this to its normalization we need 5 basic moves: slide p_9 right, p_{12} up, then followed by sliding p_{14}, p_{15} and p_5 to the left. Its normalization would take on the form:

13	2	7	4
10	3	6	8
12	9	1	11
14	15	5	

where:

$$\sigma_s = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{16} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \end{pmatrix}$$

$$\sigma_s = (11, 16, 15, 14, 13, 12)$$

3 Theorems

Cayley's Theorem implies that any group G is isomorphic to a group of permutations.

Theorem 3.1. *The group S is isomorphic to the symmetric group S_{16}*

Proof. Since $\forall$ element $a \in S$, the snapshot takes on the form:

$$a = \begin{pmatrix} p_1 & p_2 & \cdots & p_{15} & p_{16} \\ c_i & c_j & \cdots & c_k & c_l \end{pmatrix}$$

We can simply ignore the p and c notation in the element and find ourselves an element in S_{16} i.e. if $f : S \to S_{16}$ where $f(a) = b$ Then b takes on the form:

$$b = \begin{pmatrix} 1 & 2 & \cdots & 15 & 16 \\ i & j & \cdots & k & l \end{pmatrix}$$

□

Theorem 3.2. *The group of all configurations is isomorphic to the symmetric group S_{15}*

Proof. This proof is simply the same as Theorem 3.1, but we now have a fixed transposition where $\forall\ a \in S$, $a(p_{16}) = c_{16}$. Therefore, we can treat any configuration as a permutation of 15 pieces $\{p_1, p_2, \cdots, p_{15}\}$ to cells $\{c_1, c_2, \cdots, c_{15}\}$. Therefore, the group of all configurations is isomorphic to S_{15} for the same reasoning as before.

□

Theorem 3.3. *There exists a move taking a snapshot s to t if and only if there is a move taking the normalization of the former, $\sigma = \sigma_s$ to the normalization of the latter $\tau = \tau_t$.*

Proof.

$\Rightarrow$ Suppose $t = M \circ s$
For all $s \in S$, there exists a normalization σ such that $\sigma = \sigma_s$.
Let τ_t and σ_s denote the normalization for t and s respectively.
Then, $t = M_t^{-1} \circ \tau_t$ and $s = M_s^{-1} \circ \sigma_s$ where M_t and M_s denote the move moving the blank space, p_{16}, all the way to c_{16} in their corresponding snapshots.
$M_t^{-1} \circ \tau_t = M \circ M_s^{-1}\sigma_s$
Left multiply both sides by M_t
therefore $\tau_t = M_t \circ M \circ M_s^{-1} \circ \sigma_s$
The composition of $M_t \circ M \circ M_s^{-1}$ is a move that takes the normalization σ_s to τ_t.

$\Leftarrow$ Suppose $\tau_t = M \circ \sigma_s$
Where $\tau_t = M_t \circ t$ and $\sigma_s = M_s \circ s$
Then, $M_t \circ t = M \circ M_s \circ s$
Left multiply by M_t^{-1}
$\therefore t = M_t^{-1} \circ M \circ M_s \circ s$
The composition of $M_t^{-1} \circ M \circ M_s$ is a move that takes the snapshot s to t.
This completes the proof.

□

Corollary 3.3.1. Since for any normalization of a snapshot s, $\sigma_s(p_{16}) = c_{16}$, then σ_s is a configuration. As shown by Theorem 3.3, any two snapshots s and t such that $t = M \circ s$ there must also exist a move that relates their configurations. Therefore, we can solely inspect the set of configurations in order to solve the puzzle.

From now on, let's restrict ourselves to the set of all moves between configurations only, let's call this set $M(C)$.

Theorem 3.4. *Any move $M \in M(C)$ can be characterized of the form: $a = (16, i_{m-1})(i_{m-1}, i_{m-2})...(i_2, i_1)(i_1, 16)$ where i_k is a neighbor of i_{k+1}.*

Proof. Since a move M $\in M(C)$ simply moves the blank from c_{16} to c_{16} by some permutation, and any permutation is comprised of a product of transpositions (where each transposition is a basic move m), then M must always take on the form of a. □

Lemma 3.1. $M(C)$ is a subgroup of A_{15}.

In order to prove that $M(C)$ is a subgroup of A_{15}, we must show the following: $M(C)$ is a subset of A_{15}, $M(C)$ is closed under composition of elements $\in M(C)$, there exists an identity element, and finally that for any element a $\in M(C)$ there also exists an inverse $a^{-1} \in M(C)$.

Proof. For a given cell c_i, all neighboring cells accessible via a legal basic move m is always of opposite parity. Therefore, a move (not necessarily $\in M(C)$) from some cell c_i to c_k where i and k share the same parity is always divisible by two. Since a move M $\in M(C)$ move cell c_{16} to itself, it is therefore comprised of an even number of transpositions.

The identity element is trivial, since the empty move $\in M(C)$ acts as the identity element.

For a given element $a \in M(C)$, the inverse simply swaps the order in which the transpositions are performed such that if $a = (16, i_{m-1})(i_{m-1}, i_{m-2})...(i_2, i_1)(16, i_1)$ where i_k is a neighbor of i_{k+1}, then $a^{-1} = (16, i_1)(i_2, i_1)...$ $(i_{m-1}, i_{m-2})(16, i_{m-1})$ which is also even and therefore $\in M(C)$.

The composition of elements $a, b \in M(C)$ must always be even, given that the composition of even transpositions is also even. For instance, suppose a takes N_1 to N_2 by $N_1 = a \circ N_2$ and b takes N_2 to N_3 by $N_2 = b \circ N_3$. Then, the composition $a \circ b$ is a move where $N_1 = a \circ b \circ N_3$ which is also even.

Proof Complete. □

Corollary 3.4.1. Every configuration is an equivalence relation on the set of snapshots of order 16.

Proof. Since for any snapshot s we can perform a normalization $\sigma = \sigma_s$, there exists a subset of snapshots $\{s_1, s_2, ..., s_n\}$ that result in the same configuration. It is clear for a given snapshot, by moving the position of the empty cell along the wiggly trajectory there are 16 distinct snapshots that all result in that same configuration once normalized. ☐

Remark. All moves between equivalence classes can be characterized by a finite product of transpositions of the form $a = (16,15)(15,14)...(j+2,j+1)(j,i)(i,i+1)...(15,14)(16,15)$ where $a \in M(C)$.

Explanation
The generic characterization of these moves can be broken into three distinct sections; these three sections are displayed by the orange, violet, and blue in the theorem. Remember, transposition composition are from right to left, and so I will explain the setup for the characterization of moves $\in M(C)$ in the opposite order.
Suppose we want to show the move between the two following configurations:

$N_1 =$

13	2	7	4
10	3	6	8
12	9	1	11
14	15	5	

$N_2 =$

13	2	7	4
10	6	8	11
12	9	3	1
14	15	5	

Step 1: Move the blank space along the wiggly trajectory to c_{11} by $blue = (12,11)(13,12)(14,13)(15,14)(16,15)$.

$blue \circ N_1 =$

13	2	7	4
10	3	6	8
9		1	11
12	14	15	5

Step 2: Perform the basic move *up* not along the wiggle trajectory such that we enter our new set of snapshots that relate by a different normalization, as seen by configuration N_2 by $violet = (11,6)$.

$violet \circ blue \circ N_1 =$

13	2	7	4
10		6	8
9	3	1	11
12	14	15	5

Step 3: Perform the normalization that results in configuration N_2 by $orange = (16,15)(15,14)(14,13)(13,12)(12,11)$ $(11,10)(10,9)(9,8)(8,7)(7,6)$.

$N_2 = orange \circ violet \circ blue \circ N_1 =$

13	2	7	4
10	6	8	11
12	9	3	1
14	15	5	

There are a total of 9 moves $\in M(C)$ that results in the transition from one equivalence class to another. Let's call this subset ρ. See Appendix A for all moves, their inverses and their labels m_i where $1 \leq i \leq 9$.

Lemma 3.2. The group generated by the subset ρ contains all consecutive three-cycles $(k, k+1, k+2)$ in A_{15}.

$(1,2,3) = (7,6,5,4,3,2,1)^2(3,4,5)(1,2,3,4,5,6,7)^2 = m_7^2 \circ m_9^{-1} \circ m_7^{-2}$

$(2,3,4) = (7,6,5,4,3,2,1)(3,4,5)(1,2,3,4,5,6,7) = m_7 \circ m_9^{-1} \circ m_7^{-1}$

$(3,4,5) = m_9^{-1}$

$(4,5,6) = (1,2,3,4,5,6,7)(3,4,5)(7,6,5,4,3,2,1) = m_7^{-1} \circ m_9^{-1} \circ m_7$

$(5,6,7) = (1,2,3,4,5,6,7)^2(3,4,5)(7,6,5,4,3,2,1)^2 = m_7^{-2} \circ m_9^{-1} \circ m_7^2$

$(6,7,8) = (11,10,9,8,7,6,5)(7,8,9)(5,6,7,8,9,10,11) = m_4 \circ m_6^{-1} \circ m_4^{-1}$

$(7,8,9) = m_6^{-1}$

$(8,9,10) = (5,6,7,8,9,10,11)(7,8,9)(11,10,9,8,7,6,5) = m_4^{-1} \circ m_6^{-1} \circ m_4$

$(9,10,11) = (5,6,7,8,9,10,11)(7,8,9)(11,10,9,8,7,6,5) = m_4^{-2} \circ m_6^{-1} \circ m_4^2$

$$(10,11,12) = (15,14,13,12,11,10,9)(11,12,13)(9,10,11,12,13,14,15) = m_1 \circ m_3^{-1} \circ m_1^{-1}$$
$$(11,12,13) = m_3^{-1}$$
$$(12,13,14) = (9,10,11,12,13,14,15)(11,12,13)(15,14,13,12,11,10,9) = m_1^{-1} \circ m_3^{-1} \circ m_1$$
$$(13,14,15) = (9,10,11,12,13,14,15)(11,12,13)(15,14,13,12,11,10,9) = m_1^{-2} \circ m_3^{-1} \circ m_1^2$$

Theorem 3.5. *All consecutive three-cycles generate A_n for $n \geq 3$.*

Proof. We will proceed by induction. For A_3, it is trivially clear that since $A_n = \{i, (1,2,3), (1,3,2)\}$ then the subset of consecutive three-cycles (i.e. $(1,2,3)$) generates the whole set since $(1,2,3)^2 = (1,3,2)$ and $(1,2,3)^3 = i$. Suppose that A_n is generated by consecutive three-cycles. Then, we must prove that this also holds for the alternating group A_{n+1}.
Suppose there exists a permutation σ in A_{n+1}. If $\sigma(n+1) = n+1$, then σ is also in A_n since $n+1$ maps to itself and therefore it is a product of consecutive three-cycles. Alternatively, suppose $\sigma(n+1) = a$ where $a < n+1$. Then, by another permutation $\alpha(a) = n$ where α is an element in A_n (we know α exists since A_n is transitive). If $b = (n-1, n, n+1) \circ \alpha \circ \sigma$, b must be an element in A_n since $n+1$ maps to itself. Therefore, by our same reasoning as earlier, b is a product of consecutive three-cycles. By re-arranging for σ we obtain $\sigma = \alpha^{-1} \circ (n+1, n, n-1) \circ b$. Since α & b are in A_n, then it is clear that σ is generated via a composition of consecutive three-cycles.
Proof Complete. ☐

Corollary 3.5.1. $M(C) = A_n$.

Proof. Since **Lemma 3.2** and **3.5** we can deduce that since $M(C)$ contains all consecutive three-cycles that generate A_n, and in **Lemma 3.1** we had shown that $M(C)$ is a subgroup of A_n, then it must hold that $M(C) = A_n$. ☐

Theorem 3.6. *For $n \geq 3$, the three-cycles generate A_n*

Proof. Using (Conrad, 2023) we can prove the given theorem.
The identity element $e \in A_N$ is simply the product of any two three-cycle element $a \in A_n$ and its inverse.
For a non-identity element b $\in A_n$, we can write it as a product of even transpositions such that: $b = a_1 \circ a_2 \circ ... \circ a_k$ where a_i denotes a transposition and k is an even number. We will now show that every product of two transpositions $\in A_n$ is a product of two three-cycles and such b must be a product of three-cycles. There are three cases we must ensure result in a product of two three-cycles, as shown below:
Case 1: Suppose a_i and a_{i+1} are equal.
Then, the composition of these two transpositions equal the identity, which we can replace with $(1,2,3)(1,3,2)$, a product of two three-cycles.
Case 2: Suppose a_i and a_{i+1} have only one element in common.
Then, the composition of the two transpositions will take the following form: $a_i \circ a_{i+1} = (a,c)(a,d) = (a,d,c)$.
Case 3: Suppose a_i and a_{i+1} share no two elements in common.
Then, the composition of these two transpositions will take the following form: $a_i \circ a_{i+1} = (a,c)(d,e) =$
$(a,c)(c,d)(c,d)(d,e) = (c,d,a)(d,e,c)$ where $a \neq c \neq e \neq d$, a product of two three-cycles.

Since, in all three cases, we can compose all two transpositions as a product of two three-cycles then $\forall\ b \in A_n$, b can be made as a product of three-cycles (Conrad, 2023). ☐

Theorem 3.7. *A puzzle is solvable if and only if the required permutation defining the move between the normalization of the given puzzle and the initial setup is even.*

Proof.

$\Rightarrow$ Suppose a puzzle is solvable
For a puzzle to be solvable, there must exist a move M taking the given puzzle (snapshot) s to the un-shuffled initial position. Let us call this un-shuffled initial position t. Then, by **3.3**, $\exists$ a move taking the normalization of the former, σ_s, to the latter τ_t. This move must therefore exist $\in M(C)$. Since by **3.5.1** $M(C) = A_n$, the permutation between σ_s and τ_t must be even.

$\Leftarrow$ suppose the required permutation between the normalization of the given puzzle and the initial setup is even.
Therefore, the required permutation is $\in A_n = M(C)$. Then, given **3.3** there exists a move between the given puzzle s and the initial position t. The puzzle is solvable. ☐

To further concrete the material covered, let's look at some examples. Note, we will be omitting the p_i and c_k notation from the permutations that signify snapshots.
Suppose we have the following snapshot s, and we want to solve the puzzle by returning to the un-shuffled initial position, t.

2	10	3	4
1	9	6	8
5		7	12
13	14	11	15

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 5 & 4 & 2 & 1 & 12 & 7 & 10 & 8 & 6 & 3 & 15 & 9 & 13 & 14 & 16 & 11 \end{pmatrix}$$

$$s = (1, 5, 12, 9, 6, 7, 10, 3, 2, 4)(11, 15, 16)$$

We can normalize s by

$$M_s = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 16 & 11 & 12 & 13 & 14 & 15 \end{pmatrix}$$

$$M_s = (11, 16, 15, 14, 13, 12)$$

Which provides the following snapshot σ_s:

2	10	3	4
1	9	6	8
13	5	7	12
14	11	15	

Where $\sigma_s = M_s \circ s$.

Since $t = \tau_t$ where τ_t is the normalization of t, then M such that $\tau_t = M \circ \sigma_s$ must be even given **3.7**.

$$M = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 2 & 11 & 3 & 4 & 12 & 6 & 8 & 9 & 7 & 5 & 13 & 14 & 10 & 15 & 16 \end{pmatrix}$$

$$M = (3, 11, 5, 4)(6, 12, 13, 14, 10, 7)$$

$$M = (16, 15)(15, 10)(10, 7)(7, 6)(11, 6)(12, 11)(12, 5)(5, 4)(4, 3)(6, 3)(11, 6)(12, 11)(12, 13)(14, 13)(15, 14)(16, 15)$$

At this stage it is clear that M is comprised of 16 transpositions, an even number. Therefore, it is possible to solve this puzzle.

On the other hand, suppose we are dealt the initial snapshot l:

1	2	3	4
6	5	8	7
9	10	11	12
14	13	15	

Where clearly $l = \lambda_l$ where λ_l is the normalization of l. Then the move from λ_l to τ_t is given by the odd permutation:

$$M = (6, 5)(8, 7)(14, 13)$$

Where $M \notin A_{15}$ and therefore by **3.7** impossible to solve.

4 Conclusion

We now have the full picture. It is evident that the required permutation M from the *Evil Puzzle*, s, to the un-shuffled initial position t, takes the form $M = (15, 14)$ such that $t = M \circ s$. It is evident that $M \notin A_{15}$ given that M is odd, and by **3.5.1**, M therefore does not exist in $M(C)$. By **3.7**, we can finally conclude that the *Evil Puzzle* is unsolvable. Hence, Loyd would never have to pay the \$1,000 to the unsuspecting American public. By the same reason of thinking, we can generalize this result. Suppose we begin from a snapshot a and wish to move to some new snapshot b. So long as the parity of the cells containing piece p_{16} are the same, there will exist an even move between the normalization of a and b and therefore it is possible to move between the two.

5 Appendix A

$m_1 = (16,15)(15,14)(14,13)(13,12)(12,11)(11,10)(10,9)(16,9) = (15,14,13,12,11,10,9)$

$m_2 = (16,15)(15,14)(14,13)(13,12)(12,11)(11,10)(15,10)(16,15) = (14,13,12,11,10)$

$m_3 = (16,15)(15,14)(14,13)(13,12)(12,11)(14,11)(15,14)(16,15) = (13,12,11)$

$m_4 = (16,15)(15,14)(14,13)(13,12)(12,11)(11,10)(10,9)(9,8)(8,7)(7,6)(6,5)(12,5)(13,12)(14,13)(15,14)(16,15) = (11,10,9,8,7,6,5)$

$m_5 = (16,15)(15,14)(14,13)(13,12)(12,11)(11,10)(10,9)(9,8)(8,7)(7,6)(11,6)(12,11)(13,12)(14,13)(15,14)(16,15) = (10,9,8,7,6)$

$m_6 = (16,15)(15,14)(14,13)(13,12)(12,11)(11,10)(10,9)(9,8)(8,7)(10,7)(11,10)(12,11)(13,12)(14,13)(15,14)(16,15) = (9,8,7)$

$m_7 = (16,15)(15,14)(14,13)(13,12)(12,11)(11,10)(10,9)(9,8)(8,7)(7,6)(6,5)(5,4)(4,3)(3,2)(2,1)(8,1)(9,8)(10,9)(11,10)(12,11)(13,12)(14,13)(15,14)(16,15) = (7,6,5,4,3,2,1)$

$m_8 = (16,15)(15,14)(14,13)(13,12)(12,11)(11,10)(10,9)(9,8)(8,7)(7,6)(6,5)(5,4)(4,3)(3,2)(7,2)(8,7)(9,8)(10,9)(11,10)(12,11)(13,12)(14,13)(15,14)(16,15) = (6,5,4,3,2)$

$m_9 = (16,15)(15,14)(14,13)(13,12)(12,11)(11,10)(10,9)(9,8)(8,7)(7,6)(6,5)(5,4)(4,3)(6,3)(7,6)(8,7)(9,8)(10,9)(11,10)(12,11)(13,12)(14,13)(15,14)(16,15) = (5,4,3)$

$m_1^{-1} = (16,9)(10,9)(11,10)(12,11)(13,12)(14,13)(15,14)(16,15) = (9,10,11,12,13,14,15)$

$m_2^{-1} = (16,15)(15,10)(11,10)(12,11)(13,12)(14,13)(15,14)(16,15) = (10,11,12,13,14)$

$m_3^{-1} = (16,15)(15,14)(14,11)(12,11)(13,12)(14,13)(15,14)(16,15) = (11,12,13))$

$m_4^{-1} = (16,15)(15,14)(14,13)(13,12)(12,5)(6,5)(7,6)(8,7)(9,8)(10,9)(11,10)(12,11)(13,12)(14,13)(15,14)(16,15) = (5,6,7,8,9,10,11)$

$m_5^{-1} = (16,15)(15,14)(14,13)(13,12)(12,11)(11,6)(7,6)(8,7)(9,8)(10,9)(12,11)(13,12)(14,13)(15,14)(16,15) = (6,7,8,9,10)$

$m_6^{-1} = (16,15)(15,14)(14,13)(13,12)(12,11)(11,10)(10,7)(8,7)(9,8)(10,9)(11,10)(12,11)(13,12)(14,13)(15,14)(16,15) = (7,8,9)$

$m_7^{-1} = (16,15)(15,14)(14,13)(13,12)(12,11)(11,10)(10,9)(9,8)(8,1)(2,1)(3,2)(4,3)(5,4)(6,5)(7,6)(8,7)(9,8)(10,9)(11,10)(12,11)(13,12)(14,13)(15,14)(16,15) = (1,2,3,4,5,6,7)$

$m_8^{-1} = (16,15)(15,14)(14,13)(13,12)(12,11)(11,10)(10,9)(9,8)(8,7)(7,2)(3,2)(4,3)(5,4)(6,5)(7,6)(8,7)(9,8)(10,9)(11,10)(12,11)(13,12)(14,13)(15,14)(16,15) = (2,3,4,5,6)$

$m_9^{-1} = (16,15)(15,14)(14,13)(13,12)(12,11)(11,10)(10,9)(9,8)(8,7)(7,6)(6,3)(4,3)(5,4)(6,5)(7,6)(8,7)(9,8)(10,9)(11,10)(12,11)(13,12)(14,13)(15,14)(16,15) = (3,4,5)$

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